

THE RAMSEY NUMBERS FOR A TRIPLE OF LONG CYCLES

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ABSTRACT. We find the asymptotic value of the Ramsey number for a triple of long cycles, where the lengths of the cycles are large but may have different parity.

1. INTRODUCTION

If G_1, G_2, \dots, G_k are graphs, then the Ramsey number $R(G_1, \dots, G_k)$ is the smallest number N such that each coloring of the edges of the complete graph K_N on N vertices with k colors leads to a monochromatic copy of G_i in the i th color for some i , $1 \leq i \leq k$. The exact value of $R(G_1, \dots, G_k)$ is known only when all (or most) of G_i 's are either small, or of a very special kind (cf. Radziszowski's 'dynamic survey' [12]). In this paper we consider the case in which $k = 3$ and all graphs are long cycles.

The value of the Ramsey number $R(C_{m_1}, C_{m_2})$ for a pair of cycles was determined independently by Faudree and Schelp [5], and Rosta [13] (see also [8]). A few years later Erdős *et al.* [3] found the value of $R(C_{m_1}, C_{m_2}, C_{m_3})$ and $R(C_{m_1}, C_{m_2}, C_{m_3}, C_{m_4})$ when one of the cycles is much longer than the others. Bondy and Erdős [1] (see also [2]) conjectured that if m is odd, then the value of $R(C_m, C_m, C_m)$ is equal to $4m - 3$. Łuczak [11] used the Regularity Lemma to show that an 'asymptotic version' of this conjecture holds, i.e., for a large odd m we have $R(C_m, C_m, C_m) = (4 + o(1))m$. Then, Kohayakawa, Simonovits and Skokan [10] employed the Regularity Lemma to show that Bondy-Erdős' conjecture holds for large values of m . The asymptotic value of the Ramsey number for a triple of long even cycles was found by Figaj and Łuczak [6] (see Theorem 1(i) below). A similar result was proved independently by Gyárfás *et al.* [7], who also found an exact solution for a closely related problem of finding the Ramsey number for a triple of long paths of the same length.

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Below we establish the asymptotic value of the Ramsey number for a triple of long cycles in the non-diagonal case. As in the case of the pair of cycles, it turns out that the value of $R(C_{m_1}, C_{m_2}, C_{m_3})$ strongly depends on the parity of m_i 's. Thus, let $\langle x \rangle$ be the maximum odd number not larger than x and $\langle\langle x \rangle\rangle$ denote the maximum even number not larger than x . Then the main result of this paper can be stated as follows.

Theorem 1. *Let $\alpha_1, \alpha_2, \alpha_3 > 0$.*

- (i) $R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle\langle \alpha_2 n \rangle\rangle}, C_{\langle\langle \alpha_3 n \rangle\rangle}) = (0.5\alpha_1 + 0.5\alpha_2 + 0.5\alpha_3 + 0.5 \max\{\alpha_1, \alpha_2, \alpha_3\} + o(1))n$,
- (ii) $R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle\langle \alpha_2 n \rangle\rangle}, C_{\langle \alpha_3 n \rangle}) = (\max\{2\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 0.5\alpha_1 + 0.5\alpha_2 + \alpha_3\} + o(1))n$,
- (iii) $R(C_{\langle \alpha_1 n \rangle}, C_{\langle \alpha_2 n \rangle}, C_{\langle \alpha_3 n \rangle}) = (\max\{4\alpha_1, \alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_3\} + o(1))n$,
- (iv) $R(C_{\langle \alpha_1 n \rangle}, C_{\langle \alpha_2 n \rangle}, C_{\langle \alpha_3 n \rangle}) = (4 \max\{\alpha_1, \alpha_2, \alpha_3\} + o(1))n$.

Let us note that the first part of the above theorem is just a reformulation of the result of Figaj and Łuczak [6], while the forth one follows rather easily from the result of Łuczak [11] (see the end of Section 2). Thus, our main task will be to verify (ii) and (iii).

The structure of the paper is the following. In the next section we briefly sketch our approach to the problem which follows the idea introduced in [11]. It is based on a simple observation that, because of the Regularity Lemma, finding large monochromatic cycles in k -colored graphs is not much harder than finding matchings in monochromatic components in k -colored graphs. Thus, in order to show Theorem 1, one needs to prove a Ramsey-like result for matchings. To this end, in Section 3 we state two structural results characterizing graphs without large matchings. In the next section we briefly describe the way we approach this problem and give a few technical lemmas needed for our argument. Finally, in the next two parts of the paper, Sections 5 and 6, we prove the second and the third parts of Theorem 1, respectively.

All graphs considered in this paper are simple, without loops and multiple edges. For a graph $G = (V, E)$, and disjoint subsets A, B of V , by $e(A, B) = e_G(A, B)$ we mean the number of edges $\{u, v\}$ with $u \in A$ and $v \in B$. By $G[A]$ we denote the subgraph of G induced by $A \subseteq V$. Throughout the paper $d(v) = d_G(v)$ stands for the degree of a vertex $v \in V$ in a graph $G = (V, E)$, while $d(G) = 2|E|/|V|$ is the average degree of (vertices in) a graph G .

2. CYCLES AND MATCHINGS

Let us recall that the proof of Theorem 1 relies on a simple observation from [11] that finding a matching contained in a monochromatic component in a ‘cluster graph’ covering an α -fraction of vertices leads to a monochromatic cycle covering $(1 + o(1))\alpha$ vertices in the original graph. Below we state this connection in a formal way. The main technical nuisance here is that the ‘cluster graph’ used in the Regularity Lemma is not complete: it lacks a small fraction of edges which, however, can be made arbitrarily small.

In order to make our argument precise we introduce two relations $\sigma_{t,s}$ and $\tau_{t,s}$ defined in the following way. Let t, s be nonnegative integers and let a_1, \dots, a_{t+s}, c be positive real numbers. Then the relation $\sigma_{t,s}(a_1, \dots, a_{t+s}; c)$ holds if for every $\delta > 0$ there exists $\epsilon > 0$ and n_0 such that for every $n > n_0$ and any graph G on $N > (1 + \delta)cn$ vertices, with at least $N > (1 - \epsilon)\binom{N}{2}$ edges, every edge coloring of G using $t + s$ colors results in either an even cycle of length $\langle\langle a_i n \rangle\rangle$ in the i -th color, for some $i = 1, \dots, t$, or an odd cycle of length $\langle a_j n \rangle$ in the j -th color, for some $j = t + 1, \dots, t + s$.

In a similar way we define the relation $\tau_{t,s}(a_1, \dots, a_{t+s}; c)$ which holds if for every $\delta > 0$ there exists $\epsilon > 0$ and n_0 such that for every $n > n_0$ and any graph G on $N > (1 + \delta)cn$ vertices, with at least $N > (1 - \epsilon)\binom{N}{2}$ edges, every edge coloring of G using $t + s$ colors results in either a matching contained in a component of the i -th color, for some $i = 1, \dots, t$, saturating at least $a_i n$ vertices, or a matching contained in a non-bipartite component of the j -th color, for some $j = t + 1, \dots, t + s$, saturating at least $a_j n$ vertices.

Notice that if $\sigma_{t,s}(a_1, \dots, a_{t+s}; c)$ holds then

$$R(C_{\langle\langle a_1 n \rangle\rangle}, \dots, C_{\langle\langle a_t n \rangle\rangle}, C_{\langle a_{t+1} n \rangle}, \dots, C_{\langle a_{t+s} n \rangle}) \leq (c + o(1))n. \quad (1)$$

Indeed, inequality (1) says that the condition in definition of $\sigma_{t,s}$ holds for $\epsilon = 0$. The following simple fact is a straightforward consequence of the definitions of $\sigma_{t,s}$ and $\tau_{t,s}$.

Lemma 2. (i) *If $\sigma_{t,s}(a_1, \dots, a_{t+s}; c)$ then $\tau_{t,s}(a_1, \dots, a_{t+s}; c)$.*
 (ii) *If $t \leq t'$, $t + s = t' + s'$, $a_i \geq a'_i$ for $i = 1, \dots, t + s$, and $\tau_{t,s}(a_1, \dots, a_{t+s}; c)$ holds, then we have also $\tau_{t',s'}(a'_1, \dots, a'_{t+s}; c)$.* \square

The next result, crucial for our argument, states that the relation from Lemma 2(i) can be reversed.

Lemma 3. *If $\tau_{t,s}(a_1, \dots, a_{t+s}; c)$ then $\sigma_{t,s}(a_1, \dots, a_{t+s}; c)$.*

The proof of Lemma 3 is based on the Szemerédi's Regularity Lemma. Let us first recall some definition related to this result. Let $G = (V, E)$ be a graph and let A, B be disjoint subsets of V . We say that a pair (A, B) is (ϵ, G) -regular for some $\epsilon > 0$ if for every $A' \subseteq A$, $|A'| \geq \epsilon|A|$, and $B' \subseteq B$, $|B'| \geq \epsilon|B|$, we have

$$\left| \frac{e(A', B')}{|A'||B'|} - \frac{e(A, B)}{|A||B|} \right| < \epsilon.$$

A partition $\Pi = (V_i)_{i=0}^k$ of the vertex set V of G is (ϵ, k) -equitable if $|V_0| \leq \epsilon|V|$ and $|V_1| = \dots = |V_k|$. An (ϵ, k) -equitable partition $\Pi = (V_i)_{i=0}^k$ is (k, ϵ, G) -regular if at most $\epsilon \binom{k}{2}$ of the pairs (V_i, V_j) , $1 \leq i < j \leq k$, are not (ϵ, G) -regular. Szemerédi's Regularity Lemma [14] (see also [9]) states that every graph G admits an (k, ϵ, G) -regular partition for some k , where $1/\epsilon \leq k \leq K_0$, and the constant K_0 depends only on ϵ but not on the choice of G . Below we use the following general version of this result.

Lemma 4. *For every $\epsilon > 0$, k , and ℓ , there exists $K_0 = K_0(\epsilon, k_0, \ell)$ such that the following holds. For all graphs, G_1, G_2, \dots, G_ℓ , with $V(G_1) = V(G_2) = \dots = V(G_\ell) = V$ and $|V| \geq k_0$, there exists a partition $\Pi = (V_0, V_1, \dots, V_k)$ of V such that $k_0 \leq k \leq K_0$ and Π is (k, ϵ, G_r) -regular for all $r = 1, 2, \dots, \ell$. \square*

We shall also need the following simple property of (ϵ, G) -regular pairs (for a similar result see, for instance, [11]).

Lemma 5. *Let $T \geq 2$, $0 < \epsilon < 1/(100T)$, and let $G = (V, E)$ be a bipartite graph with bipartition $\{V_1, V_2\}$ such that $|V_1| = |V_2| = n > 10T\epsilon^{-2}$. Furthermore, let $e(V_1, V_2) \geq |V_1||V_2|/T$ and let the pair (V_1, V_2) be (ϵ, G) -regular. Then, for every ℓ , $1 \leq \ell \leq n - 5\epsilon n$, and every pair of vertices $v' \in V_1$, $v'' \in V_2$, where $d(v'), d(v'') \geq n/(5T)$, G contains a path of length $2\ell + 1$ connecting v' and v'' . \square*

Now we can show Lemma 3.

Proof of Lemma 3. Let $a_i \geq a'_i$ for all $i = 1, \dots, t + s$ and let G be a graph with $N > (1 + \delta)cn$ vertices and at least $(1 - \epsilon_\tau^4(\delta/2)) \binom{N}{2}$ edges, where $\epsilon_\tau(\delta/2)$ is a constant defined as in relation $\tau_{t,s}(a_1, \dots, a_{t+s}; c)$. Let us assume that $(G_1, G_2, \dots, G_{t+s})$ is a $t + s$ -coloring of the edges of G .

Now let $\epsilon = \min\{\delta/4, \epsilon_\tau^4(\delta/2)\}$. Apply Lemma 4 to find a partition $\Pi = \{V_0, V_1, \dots, V_k\}$ of vertices of G such that $1/\epsilon_\tau(\delta/2) \leq k \leq K_0$ and Π is (k, ϵ, G_ℓ) -regular for all $\ell = 1, 2, \dots, t + s$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph with vertex set $\mathcal{V} = \{V_1, V_2, \dots, V_k\}$ and

$$\mathcal{E} = \{\{V_i, V_j\} : (V_i, V_j) \text{ is } (\epsilon, G_\ell)\text{-regular for } \ell = 1, 2, \dots, t+s, \\ \text{and } d_G(V_i, V_j) \geq \frac{1}{2}|V_i||V_j|\}.$$

Then $|\mathcal{E}| \geq (1 - \epsilon_\tau^2(\delta/2))\binom{k}{2}$. Construct a $t+s$ -coloring $(\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_{t+s})$ of \mathcal{E} by coloring an edge $\{V_i, V_j\}$ with the lexicographically first color ℓ for which

$$e_{G_\ell}(V_i, V_j) \geq \frac{|V_i||V_j|}{2(t+s)}.$$

Let $k = (1 + \delta/2)ck'$. Then there is a color ℓ_0 , $1 \leq \ell_0 \leq t+s$, such that \mathcal{G} contains a matching $\mathcal{M} = \{e_1, e_2, \dots, e_q\}$ saturating $2q \geq a_{\ell_0}k'$ vertices of G in the monochromatic component of \mathcal{G} in the ℓ_0 th color; furthermore, if $\ell_0 \geq t+1$, then this component is non-bipartite. Note that since for every i , $i = 1, 2, \dots, k$, we have

$$|V_i| \geq \frac{(1 + \delta)cn - \epsilon n}{k} = \frac{(1 + \delta)cn - \epsilon n}{(1 + \delta/2)ck'} \geq (1 + \delta/4)\frac{n}{k'},$$

the total number of vertices of G contained in the sets V_i saturated by \mathcal{M} is at least

$$a_{\ell_0}k'|V_i| \geq a_{\ell_0}k'(1 + \delta/4)\frac{n}{k'} = (1 + \delta/4)a_{\ell_0}n \geq (1 + 6\epsilon)a_{\ell_0}n. \quad (2)$$

We shall show that the subgraph spanned in G by these vertices contains a monochromatic cycle in the ℓ_0 th color on precisely $\langle\langle a_{\ell_0}n \rangle\rangle$ vertices if $1 \leq \ell_0 \leq t$, and with $\langle a_{\ell_0}n \rangle$ vertices if $t+1 \leq \ell_0 \leq t+s$.

Let us assume first $t+1 \leq \ell_0 \leq t+s$. Let us recall that \mathcal{G} contains a monochromatic component \mathcal{F} in the ℓ_0 th color which contains a matching $\mathcal{M} = \{e_1, e_2, \dots, e_q\}$, $2q \geq a_{\ell_0}k'$, and an odd cycle $\mathcal{C} = W_1W_2 \dots W_p\hat{V}_1$. Observe that \mathcal{F} contains a closed walk $\mathcal{W} = \hat{V}_1\hat{V}_2 \dots \hat{V}_p\hat{V}_1$ of an odd length, containing all edges of \mathcal{M} . Indeed, it is easy to see that the minimum connected subgraph of \mathcal{F} which contains all edges of \mathcal{M} and a vertex \hat{V}_1 of \mathcal{C} is a tree \mathcal{T} . Since clearly there is an (even) walk \mathcal{W}' which traverses each edge of \mathcal{T} precisely two times, in order to get \mathcal{W} it is enough to enlarge \mathcal{W}' by edges of \mathcal{C} .

Now, using elementary properties of (ϵ, G) -regular pairs, it is easy to find in G an odd cycle $C = \hat{v}_1\hat{v}_2 \dots \hat{v}_p\hat{v}_1$ such that $\hat{v}_i \in \hat{V}_i$ for $i = 1, \dots, p$, and whenever the pair $(\hat{V}_i, \hat{V}_{i+1})$ belongs to a matching \mathcal{M} then both $d_{G_{\ell_0}}(\{\hat{v}_i\}, \hat{V}_{i+1}) \geq \frac{|\hat{V}_{i+1}|}{4(t+s)}$, and $d_{G_{\ell_0}}(\hat{V}_i, \{\hat{v}_{i+1}\}) \geq \frac{|\hat{V}_i|}{4(t+s)}$. Now we can apply Lemma 5 and replace all edges $\{\hat{v}_i, \hat{v}_{i+1}\}$, such that

$\{\hat{V}_i, \hat{V}_{i+1}\}$ belongs to \mathcal{M} , by long paths not containing any other vertices of C . Since, by (2), the number of vertices of G contained in V_i 's saturated by \mathcal{M} is larger than $(1 + 6\epsilon)a_{\ell_0}n$, we can do it in such a way that the resulting monochromatic cycle in the ℓ_0 th color has length $\langle a_{\ell_0}n \rangle$.

If $1 \leq \ell_0 \leq t$ then the argument is basically the same. Here we start with a closed walk \mathcal{W} in \mathcal{G} of an even length which contains all edges of \mathcal{M} and, based on \mathcal{W} , we construct an even cycle C in G . Then, as in the previous case, one can use Lemma 5 to enlarge C to a cycle of length $\langle\langle a_{\ell_0}n \rangle\rangle$. \square

From Lemma 2 and 3 we get the following corollary.

Corollary 6. *If $t \leq t'$, $t + s = t' + s'$ and $a_i \geq a'_i$ for $i = 1, \dots, t + s$ and $\sigma_{t,s}(a_1, \dots, a_{t+s}; c)$ holds then we have $\sigma_{t',s'}(a'_1, \dots, a'_{t+s}; c)$. \square*

Let us comment that, unlike the relation $\sigma_{t,s}(a_1, \dots, a_{t+s})$, we do not know any simple proof that the Ramsey number for cycles is monotone, e.g., although most certainly we have

$$R(C_{20}, C_{20}, C_{20}) \leq R(C_{22}, C_{20}, C_{20}) \leq R(C_{21}, C_{20}, C_{20})$$

it is by no means clear how to verify it directly (i.e., without estimating the Ramsey numbers above). However, using Corollary 6, we can deduce Theorem 1(iv) from Łuczak's result on $R(C_n, C_n, C_n)$.

Proof of Theorem 1(iv). Let us assume that $\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0$. Łuczak [11] showed that $\sigma_{0,3}(\alpha_1, \alpha_1, \alpha_1; 4\alpha_1)$ holds. Thus, by Corollary 6, $\sigma_{0,3}(\alpha_1, \alpha_2, \alpha_3; 4\alpha_1)$ holds as well and, consequently,

$$R(C_{\langle \alpha_1 n \rangle}, C_{\langle \alpha_2 n \rangle}, C_{\langle \alpha_3 n \rangle}) \leq (4\alpha_1 + o(1))n.$$

In order to show the lower bound for $R(C_{\langle \alpha_1 n \rangle}, C_{\langle \alpha_2 n \rangle}, C_{\langle \alpha_3 n \rangle})$ consider the following coloring of the complete graph K_N on $N = 4\langle \alpha_1 n \rangle - 4$ vertices. Split the vertices of K_N into four equal parts V_1, V_2, V_3 , and V_4 . Color the edges inside each of V_i 's with the first color, the edges in pairs $(V_1, V_2), (V_2, V_3), (V_3, V_4)$ with the second one, and the edges in pairs $(V_1, V_3), (V_2, V_4), (V_1, V_4)$ with the third color. Clearly in this coloring we have no monochromatic cycles longer than $\langle \alpha_1 n \rangle - 1$ in the first color, and no odd cycles in either the second or third color. Hence,

$$R(C_{\langle \alpha_1 n \rangle}, C_{\langle \alpha_2 n \rangle}, C_{\langle \alpha_3 n \rangle}) \geq 4\langle \alpha_1 n \rangle - 3,$$

and Theorem 1(iv) follows. \square

3. TWO STRUCTURAL RESULTS

This short section consists of two simple results on the structure of graphs without large matchings.

Let us start with the following consequence of Tutte's theorem observed in [6]. Since it is crucial for our approach we recall its proof here for the completeness of the argument.

Lemma 7. *If a graph $G = (V, E)$ contains no matchings saturating at least n vertices, then there exists a partition $\{S, T, U\}$ of V such that:*

- (i) *the subgraph induced in G by T has maximum degree at most $\sqrt{|V|} - 1$,*
- (ii) *there are no edges between the sets T and U ,*
- (iii) *$|U| + 2|S| < n + \sqrt{|V|}$.*

Proof. From Tutte's theorem, if a graph $G = (V, E)$ contains no matchings saturating at least n vertices, then there exists a subset S such that the number of odd components in a graph $G[V \setminus S]$ is larger than $|V| + |S| - n$. Split the set of these components into two parts: those with at most $\sqrt{|V|}$ vertices and those larger than $\sqrt{|V|}$. The set of vertices which belong to the components from the former family we denote by T , the set of vertices of the component from the latter one by U . Then, for such a partition $V = S \cup T \cup U$, (i) and (ii) clearly hold. Moreover, since there are fewer than $\sqrt{|V|}$ components larger than $\sqrt{|V|}$, Tutte's condition gives

$$|T| > |V| + |S| - n - \sqrt{|V|},$$

so that

$$|U| = |V| - |S| - |T| < n + \sqrt{|V|} - 2|S|,$$

which gives (iii). \square

Graphs without a large matching contained in a non-bipartite component have a rather simple characterization as well (cf. Łuczak [11]). Let us recall first a classical result of Erdős and Gallai [4].

Lemma 8. *Each graph with n vertices and at least $(m-1)(n-1)/2 + 1$ edges, where $3 \leq m \leq n$, contains a cycle of length at least m . In particular, it contains a component with a matching saturating at least $m-1$ vertices.* \square

Now our second structural lemma can be stated as follows.

Lemma 9. *If no non-bipartite component of a graph $G = (V, E)$ on n vertices contains a matching saturating at least αn vertices, then there exists a partition $V = V' \cup V''$ of V such that:*

- (i) G contains no edges between sets V' and V'' ,
- (ii) the graph $G' = G[V']$ induced in G by V' is bipartite,
- (iii) the graph $G'' = G[V'']$ induced in G by V'' contains at most $0.5\alpha n|V(G'')|$ edges.

Proof. Denote by H_1, \dots, H_r components of G . Let V' consist of the vertices of all the components which are bipartite and $V'' = V \setminus V'$. Then, (i) and (ii) clearly hold. Note also that if any non-bipartite component H_i has average degree larger than αn , then, by Erdős-Gallai theorem, it contains a cycle longer than αn and thus also a matching of size at least αn contradicting our assumption. Thus, every component of G'' has average degree at most αn and (iii) follows. \square

4. THE FIRST LOOK AT THE MATCHING PROBLEM

Let us recall that, in order to show Theorem 1, it is enough to verify the property $\tau_{t,s}(\alpha_1, \alpha_2, \alpha_3; c)$ for appropriately chosen $t, s, t + s = 3$, and $c = c(\alpha_1, \alpha_2, \alpha_3)$, i.e., we need to prove that in every sufficiently large three-colored ‘nearly complete’ graph G we can find a monochromatic component containing a large matching. Lemmas 7 and 9 suggest the following approach. Suppose that a component $F = (V_F, E_F)$ of the subgraph G_1 of G induced by the first color contains no large matchings. Then, using Lemma 7, one can decompose the set vertices V_F of F into three sets S, T, U . Delete from F all vertices from S . Then in the remaining graph $H = F[T \cup U]$, all edges joining T and U , as well as all but a negligible fraction of edges contained in T , are colored with either the second or the third color. Consequently, to study matchings in three-colored ‘nearly complete’ graphs, one should first study matchings in two-colored ‘nearly complete’ graphs with ‘holes’ (in our case the hole is the set U).

Lemma 9 suggests a similar approach. Suppose that in a ‘nearly complete’ three-colored $G = (V, E)$ on n vertices the graph G_1 induced by the first color has average degree $d(G_1) = \rho n > \alpha_1 n$ yet it contains no non-bipartite component with a matching saturating at least $\alpha_1 n$ vertices. Then, there is a partition of the set of vertices of G_3 in the form $V = W_1 \cup W_2 \cup R$, where $\{W_1, W_2\}$ is a bipartition of G' , R is the set of vertices of G'' , and G' and G'' are the graphs described in Lemma 9. Note that since $d(G'') \leq \alpha_1 n$, so we must have $d(G') \geq \rho n$, and so the larger of the sets W_1, W_2 , say W_1 , must have at least $\rho(n)$ vertices and $|R| \leq (1 - 2\rho)n$. Thus, in this case, the graph $F = G[W_1 \cup R]$ is a ‘nearly complete’ graph with nearly all of the edges, except those contained in the hole R , colored with just two colors.

Since our proof of Theorem 1 is based on the above idea here we state two results, Lemmas 12 and 13, which determine the size of the largest matchings when the edges of a ‘nearly complete graph with a hole’ are colored with two colors. We begin however with two technical results from [6] we state without proofs: the first one characterizes matchings in a ‘nearly complete’ bipartite graphs, the second one describes matchings in ‘nearly complete’ bipartite graphs with two holes.

Lemma 10. *Let $G = (V, E)$ be a bipartite graph with bipartition $\{V_1, V_2\}$, where $|V_1| \geq |V_2|$, and at least $(1 - \epsilon)|V_1||V_2|$ edges, where $0 < \epsilon < 0.01$. Then there is a component in G of at least $(1 - 3\epsilon)(|V_1| + |V_2|)$ vertices which contains a matching of cardinality at least $(1 - 3\epsilon)|V_2|$.* \square

Lemma 11. *Let $0 \leq \nu_1 \leq \nu_2 \leq 1$, $0 < \epsilon < 0.01\nu_1$, $N \geq 4/\epsilon$, and let U_1, U_2 be two, not necessarily disjoint, subsets of $[N] = \{1, 2, \dots, N\}$ of $\nu_1 N$ and $\nu_2 N$ vertices respectively. Let $G = ([N], E)$ be a graph obtained from the complete graph on the set $[N]$ vertices by removing all edges contained in U_i , $i = 1, 2$, and, possibly, at most $\epsilon^3 \binom{N}{2}$ other edges. Then G contains a component with a matching saturating at least:*

- (i) $(1 - 5\epsilon)N$ vertices if $|U_2| \leq N/2$;
- (ii) $(2 - 7\epsilon)N - 2|U_2|$ vertices if $|U_2| \geq N/2$. \square

Before we state and prove two main results of this section on matchings in two-colored ‘nearly complete’ graphs with holes, let us make a simple observation we shall often use in the proof. Suppose that a graph $G_W = (V, E)$ is obtained from the complete graph with vertex set V by removing all edges contained in $W \subseteq V$. Let us color edges of G_W by two colors, and let G_1, G_2 be spanning subgraphs of G induced by the first and the second color respectively. Then, either one of these graphs is connected, or there is a partition of $W = W_1 \cup W_2$ into two non-empty sets W_1, W_2 such that all edges with one end in W_i , $i = 1, 2$, are colored with the i th color.

Lemma 12. *For every $\alpha, \beta > 0$, $\nu \geq 0$, $\max\{\alpha, \beta, \nu\} = 1$, and $0 < \epsilon < 0.01 \min\{\alpha, \beta\}$, there exists n_0 , such that for every $n > n_0$ the following holds.*

Let $G = (V, E)$ be a graph obtained from the complete graph on

$$N = (0.5\alpha + 0.5\beta + \max\{\nu, 0.5\alpha, 0.5\beta\} + 3\sqrt{\epsilon})n$$

vertices by removing all edges contained in a subset $W \subseteq V$ of size νN and no more than $\epsilon^3 n^2$ other edges. Then, every coloring of the edges of G with two colors leads to either a monochromatic component colored

with the first color containing a matching saturating at least $(\alpha + \epsilon)n$ vertices, or a monochromatic component of the second color containing a matching saturating at least $(\beta + \epsilon)n$ vertices.

Proof. Let us consider a two-coloring of edges of a graph G which fulfills the assumption of the lemma, and denote graphs induced by the edges of the first and the second colors by G_1 and G_2 respectively. Let F denote the largest monochromatic component in this coloring. Without loss of generality we can assume that F is colored with the first color. We consider two following cases.

Case 1. $|F| \geq N - \sqrt{\epsilon}n$.

Let us assume that F contains no matching saturating $(\alpha + \epsilon)n$ vertices. Then one can use Lemma 7 to find a partition of the set of vertices of F into sets S, T, U such that there are no edges of the first color between T and U , there are at most $\sqrt{N}|T|$ edges of the first color contained in T , and furthermore

$$2|S| + |U| \leq \alpha n + \epsilon n + \sqrt{N}. \quad (3)$$

Now let us consider the graph $G' = G_2[T \cup U]$. We shall show that it contains a component with a matching saturating at least $(\beta + \epsilon)n$ vertices. Since G' is a ‘nearly complete’ graph on $|T| + |U| \geq N - \sqrt{\epsilon}N - |S| \geq N - 0.5\alpha n - 2\sqrt{\epsilon}n$. with two holes, U and W , we apply Lemma 11. Thus, if $|W|, |U| \leq (N - |S|)/2$, Lemma 11(i) implies that there exists a component of the second color with a matching saturating at least

$$\begin{aligned} N - 0.5\alpha n - 2\sqrt{\epsilon}n - 5\epsilon N \\ \geq 0.5\beta n + \max\{\nu, 0.5\alpha, 0.5\beta\} + 3\sqrt{\epsilon}n - 5\epsilon n - 2\sqrt{\epsilon}n \\ \geq \beta n + \epsilon n, \end{aligned} \quad (4)$$

vertices. In the case in which $\max\{|U|, |W|\} \geq (N - |S|)/2$, from Lemma 11(ii) we infer that there exists a component of the second color with a matching saturating at least

$$\begin{aligned} 2N - 4\sqrt{\epsilon}n - 7\epsilon N - 2|S| - 2\max\{|U|, |W|\} \\ \geq 2N - 2|S| - 2\max\{|U|, |W|\} - 5\sqrt{\epsilon}N \end{aligned} \quad (5)$$

vertices. Thus, if $|W| \geq |U|$, then using (3) we can estimate the right hand side of (5) by

$$\begin{aligned} 2N - 2|S| - 2|W| - 5\sqrt{\epsilon}N &\geq 2N - \alpha n - \epsilon n - \sqrt{N} - 2\nu N - 5\epsilon N \\ &\geq \beta n + \sqrt{\epsilon}n - 2\epsilon N > \beta n + \epsilon n, \end{aligned} \quad (6)$$

while for $|U| \geq |W|$ (3) gives

$$\begin{aligned} 2N - 2|S| - 2|U| - 5\sqrt{\epsilon}N &\geq 2N - 2\alpha n - 2\epsilon n - 2\sqrt{N} - 5\epsilon N \\ &\geq \beta n + \sqrt{\epsilon}n - 6\epsilon N > \beta n + \epsilon n. \end{aligned} \quad (7)$$

This completes the proof in this case.

Case 2. $|F| < N - \sqrt{\epsilon}n$.

As we have already noticed in the remark preceding the statement of the lemma, every two-coloring which does not lead to a large monochromatic component must have a rather special structure. Thus, let us denote by W_1 the set of vertices w_1 of W such that all but at most ϵn edges adjacent to w_1 are colored with the first color, by W_2 the set of vertices w_2 of W such that all but at most ϵn edges adjacent to w_2 are colored with the second color, and $W_0 = W \setminus (W_1 \cup W_2)$. Since in the graph G lacks at most $\epsilon^3 n^2$ edges joining W with $V \setminus W$ we must have $|W_0| \leq \epsilon n$. Furthermore, $|F| \leq N - \sqrt{\epsilon}n$ implies that $\max\{|W_1|, |W_2|\} \geq 0.5\sqrt{\epsilon}n$.

Let us set $|W_1| = \alpha'n$, $|W_2| = \beta'n$. Note that $|V \setminus W| \geq \max\{0.5\alpha, 0.5\beta\} + 3\sqrt{\epsilon}n$, so if either $\alpha' \geq 0.5\alpha + 7\epsilon$ or $\beta' \geq 0.5\beta + 7\epsilon$, then we are done by Lemma 10. More generally, if the graph $H = G[V \setminus W]$ contains either a monochromatic component in the first color with a matching saturating at least $\alpha''n = \alpha n - 2\alpha'n + 15\epsilon n$ vertices, or a monochromatic component in the second color with a matching saturating at least $\beta''n = \beta n - 2\beta'n + 15\epsilon n$ vertices, the assertion follows as well. Indeed, observe first that because $|W_1|, |W_2| > 0$, all vertices of H except at most $2\epsilon n$ belong to the component of the first color and, in the same way, there are at most $2\epsilon n$ vertices of H which do not belong to the large component of the second color. Thus, we can first find a large monochromatic matching in the large component of the i th color, $i = 1, 2$, and then match unsaturated vertices of this component to vertices of W_i using Lemma 10.

Now note that every two coloring of a ‘nearly complete’ graph leads to a large monochromatic component in one of the colors, so the Case 1 considered above covers all cases in which $\nu = 0$. Furthermore,

$$\begin{aligned} 0.5\alpha'' + 0.5\beta'' + \max\{0.5\alpha'', 0.5\beta''\} + 3\sqrt{\epsilon} &\leq 0.5\alpha + 0.5\beta + \max\{0.5\alpha, 0.5\beta\} - 0.5\sqrt{\epsilon} - \nu + 20\epsilon + 3\sqrt{\epsilon} \\ &\leq 0.5\alpha + 0.5\beta + \max\{\nu, 0.5\alpha, 0.5\beta\} - \nu + 2.9\sqrt{\epsilon} < N - \nu. \end{aligned}$$

Thus, Case 1 we have just proved implies that H contains either a large component in the first color with a matching saturating at least

$\alpha''n$ vertices, or a monochromatic component in the second color with a matching saturating at least $\beta''n$ vertices, and the assertion follows. \square

If we wish to have one of the matching in a non-bipartite monochromatic component, then the condition becomes slightly more complicated.

Lemma 13. *Let $\alpha, \beta > 0$, $\nu \geq 0$, $\max\{\alpha, \beta, \nu\} = 1$, $0 < \epsilon < 0.01 \min\{\alpha, \beta\}$, and let*

$$\xi = \xi(\alpha, \beta, \nu) = \max \left\{ 0.5\alpha + 0.5\beta + \max\{0.5\alpha, 0.5\beta, \nu\}, \right. \\ \left. 1.5\alpha + \max\{0.5\alpha, \nu\} \right\}. \quad (8)$$

Then, there exists n_0 , such that for every $n > n_0$ the following holds.

Let $G = (V, E)$ be a graph obtained from the complete graph on $N = (\xi + 5\sqrt{\epsilon})n$ vertices by removing all edges contained in a subset $W \subseteq V$ of size νN and no more than $\epsilon^3 n^2$ other edges. Then, every coloring of the edges of G with two colors leads to either a monochromatic component colored with the first color containing a matching saturating at least $(\alpha + \epsilon)n$ vertices, or a non-bipartite monochromatic component of the second color containing a matching saturating at least $(\beta + \epsilon)n$ vertices.

Proof. Consider a two-coloring of edges of a graph $G = (V, E)$ which fulfills the assumption of the lemma and let G_i , $i = 1, 2$, denote the graph spanned by edges of the i th color. Since

$$\xi(\alpha, \beta, \nu) \geq 0.5\alpha + 0.5\beta + \max\{0.5\alpha, 0.5\beta, \nu\},$$

from Lemma 12 it follows that either there exists a component of G_1 which contains a matching saturating at least $(\alpha + \epsilon)n$ vertices, or there exists a component F_2 in G_2 which contains a matching saturating at least $(\beta + \epsilon)n$ vertices. Thus, the assertion follows unless the component F_2 is bipartite. Hence, we shall assume that F_2 is bipartite with bipartition $\{Z_1, Z_2\}$ and split the proof into the following two cases.

Case 1. $|F_2| \geq N - \sqrt{\epsilon}n$.

Since in this case

$$|Z_1| + |Z_2| \geq (1.5\alpha + \max\{0.5\alpha, \nu\} + 4\sqrt{\epsilon})n,$$

for some $i_0 = 1, 2$, we have both $|Z_{i_0}| \geq (\alpha + 2\sqrt{\epsilon})n$ and $|Z_{i_0} \setminus W| \geq (0.5\alpha + \sqrt{\epsilon})n$. But then, due to Lemma 10, the graph $G_1[Z_{i_0}]$ contains a component with a matching saturating at least $(\alpha + \epsilon)n$ vertices.

Case 2. $|F_2| < N - \sqrt{\epsilon}n$.

Let F_1 denote the largest component of G_1 . Let us consider first the case when $|F_1| < N - \sqrt{\epsilon}n$. Then, by the remark preceding the statement of Lemma 12, all but at most $\sqrt{\epsilon}n$ vertices of $V \setminus W$ are contained in one of the sets of the bipartition of F_2 , say, in Z_1 . But then all edges of G with both ends in Z_1 are colored with the first color and $|Z_1| \geq (3\alpha/2 + \sqrt{\epsilon})n$. Consequently, by Lemma 10, there is a component in G_1 with a matching saturating at least $(\alpha n + \epsilon)n$ vertices and the assertion follows.

Thus, we may and shall assume that the largest component $F_1 = (V_1, E_1)$ of G_1 has at least $N - \sqrt{\epsilon}n$ vertices. Suppose that it contains no matchings saturating at least $(\alpha + \epsilon)n$ vertices. Let S, T, U be the sets whose existence is assured by Lemma 7; in particular we have

$$2|S| + |U| \leq \alpha n + \epsilon n + \sqrt{N}. \quad (9)$$

Note that the subgraph $H_2 = G_2[T \cup U]$ contains a component F'_2 with a matching saturating at least $(\beta + 1.1\epsilon)n$ vertices. Indeed, consider graph \hat{G} with the same set of vertices as F_1 , obtained from G by deleting all edges of G_1 with both ends in T . Then \hat{G} fulfills assumptions of Lemma 12 with $\epsilon' = 1.1\epsilon$. On the other hand, if we color with the first color all edges of \hat{G} which have either one end in S , or both ends in U , we create no matching in this color saturating more than $(\alpha + 1.1\epsilon)n$ vertices. Hence, by Lemma 12, there must be component F'_2 in the second color which contains a matching saturating at least $(\beta + 1.1\epsilon)n$ vertices, and, because of our construction, $F'_2 \subseteq H_2$.

If F'_2 is non-bipartite we are done, so let us assume that F'_2 is bipartite. Since H_2 contains all edges of G joining T and U and all but $|T|\sqrt{N}$ edges contained in T , it is easy to see that if a graph $G[T]$ contains a component of size, say, $10\epsilon n$, then all but at most $\sqrt{\epsilon}n$ vertices of H_2 lie in the same giant component which clearly is not bipartite. Thus, because of Lemma 10, in order to keep F'_2 bipartite the set T cannot be much larger than the hole W , i.e.

$$|T| \leq |W| + 10\epsilon n \leq (\nu + 10\epsilon)n. \quad (10)$$

However, from (9) and (10) it follows that

$$\begin{aligned} |F_1| &= |T| + |U| + |S| \leq (\alpha + \epsilon + \nu + 10\epsilon)n + \sqrt{N} \\ &\leq (\alpha + \nu + 2\sqrt{\epsilon})n < (1.5\alpha + \nu + 3\sqrt{\epsilon})n \\ &< N - \sqrt{\epsilon}n, \end{aligned}$$

while as, we have seen, $|F_1| > N - \sqrt{\epsilon}n$. Thus, the component F'_2 is non-bipartite and the assertion follows. \square

Remark. It is easy to construct colorings which shows that the estimates given by Lemmas 12 and 13 are, up to epsilon terms, best possible.

5. TRIPLE OF CYCLES: ONE ODD, TWO EVEN

In this part of the paper we prove Theorem 1(ii), i.e., we estimate the Ramsey number for three long cycles, in which one is odd and the other two have even length. Let us start however with the following consequence of Lemma 12.

Lemma 14. *Let $\alpha_1 \geq \alpha_2 > 0$, $0 < \epsilon < 0.01\alpha_2$ and let $G = (V, E)$ be a graph obtained from the complete graph on $|V| \geq (2\alpha_1 + \alpha_2 + 9\sqrt{\epsilon})n$ vertices by deleting at most $|E| \leq \epsilon^4 n^2$ of its edges. Then there exists n_0 such that for every $n \geq n_0$ the following holds.*

Let us suppose that the edges of G are colored with three colors which spans graphs G_1 , G_2 , and G_3 , and that the graph G' which is a union of the bipartite components of G_3 has at least $(1.5\alpha_1 + 0.5\alpha_2 + 8\sqrt{\epsilon})n$ vertices. Then, there exists either a monochromatic component of the first color which contains a matching saturating at least $(\alpha_1 + \epsilon)n$ vertices, or a monochromatic component of the second color with a matching saturating at least $(\alpha_2 + \epsilon)n$ vertices.

Proof. Observe first that it is enough to prove the lemma for ‘nearly complete’ graphs $G = (V, E)$ with precisely $|V| = (2\alpha_1 + \alpha_2 + 5\sqrt{\epsilon})n$ vertices. Let G_1 , G_2 , G_3 denote the graphs spanned in G by the first, the second, and the third color, respectively. Furthermore, let G' denote the bipartite graph with bipartition $\{X, Y\}$, where $|X| \geq |Y|$, which is the union of all bipartite components of G_3 . Let us consider a subgraph $H = (\hat{V}, \hat{E})$ of G whose vertex set is the set $V \setminus Y$ and edges are all edges of G which are colored with either the first or the second color. Thus, H is a ‘nearly complete’ graph on $|\hat{V}| = |V \setminus Y|$ vertices with a hole $W = V \setminus (X \cup Y)$ of size $|W| = \nu n$. Thus, to complete the proof we need to verify if the assumptions of Lemma 12 hold for two-colored H . To this end note that

$$\nu \leq 0.5\alpha_1 + 0.5\alpha_2 + \sqrt{\epsilon},$$

and so

$$0.5\nu + 0.5\alpha_1 \geq \nu - 0.5\sqrt{\epsilon}.$$

Thus, for the number of vertices of H we get

$$\begin{aligned} |\hat{V}| &\geq \frac{(2\alpha_1 + \alpha_2 + 9\sqrt{\epsilon})n - \nu n}{2} + \nu n \\ &\geq (0.5\alpha_1 + 0.5\alpha_2 + 0.5\nu + 0.5\alpha_1 + 4.5\sqrt{\epsilon})n \\ &\geq (0.5\alpha_1 + 0.5\alpha_2 + \max\{\nu, 0.5\alpha_1, 0.5\alpha_2\} + 4\sqrt{\epsilon})n, \end{aligned}$$

and the assertion follows from Lemma 12. \square

Now we can find the asymptotic value of $R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle})$.

Proof of Theorem 1(ii). Let us first estimate $R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle})$ from above. From Lemma 3 it follows that to prove the upper bound in Theorem 1(ii) it is enough to verify that for $\alpha_1, \alpha_2, \alpha_3 > 0$, $\alpha_1 \geq \alpha_2$, and $c = \max\{2\alpha_1 + \alpha_2, 0.5\alpha_1 + 0.5\alpha_2 + \alpha_3\}$, we have $\tau_{2,1}(\alpha_1, \alpha_2, \alpha_3; c)$ holds. Observe that we may and shall assume that $\max\{\alpha_1, \alpha_2, \alpha_3\} = 1$.

Let $\epsilon > 0$ and let $G = (V, E)$ be a graph obtained from the complete graph on $N = (c + 10\sqrt{\epsilon})n$ vertices by removing at most $|E| \geq \epsilon^5 n^2$ edges. Let us color edges of G with three colors and denote the subgraph spanned by the i th color by G_i , $i = 1, 2, 3$. Let us consider the two following cases.

Case 1. $\alpha_1 \geq \alpha_3$.

Then the number of vertices in G is $|V| = (2\alpha_1 + \alpha_2 + 10\sqrt{\epsilon})n$. If the average density $d(G_3)$ of G_3 is smaller than $(\alpha_1 + 8\sqrt{\epsilon})n$, then either $d(G_1) \geq (\alpha_1 + \epsilon)n$, or $d(G_2) \geq (\alpha_2 + \epsilon)n$ and the assertion follows from Lemma 8. Thus, let us consider the case $d(G_3) > (\alpha_1 + 8\sqrt{\epsilon})n$. Assume that G_3 contains no component with a matching saturated as least $\alpha_3 n$ vertices and let G' and G'' be two subgraphs of G_3 whose existence is assured by Lemma 9. Note that

$$d(G'') \leq \alpha_3 n < (\alpha_1 + 8\sqrt{\epsilon})n < d(G_3)$$

and so

$$d(G') \geq d(G_3) > (\alpha_1 + 8\sqrt{\epsilon})n. \quad (11)$$

Since each bipartite graph with the average degree m must have at least $2m$ vertices, from (11) we infer that G' contains at least

$$2\alpha_1 + 16\sqrt{\epsilon} > 1.5\alpha_1 + 0.5\alpha_2 + 8\sqrt{\epsilon}$$

vertices. Now the existence of a monochromatic component with large matching in one of the first two colors follows from Lemma 14.

Case 2. $\alpha_1 \leq \alpha_3$.

In this case G has $|V| = (0.5\alpha_1 + 0.5\alpha_2 + \alpha_3 + 10\sqrt{\epsilon})n$ vertices. Figaj and Łuczak [6] (cf. Theorem 1(i)) showed that then either there exists i_0 , $i_0 = 1, 2$, such that G_{i_0} contains a monochromatic component in the

i_0 th color with a matching saturating at least $\alpha_{i_0}n$ vertices, so we are done, or there exists a component G' in the third color which contains a matching saturating at least

$$\begin{aligned} |V| - (0.5\alpha_1 + 0.5\alpha_2 + \sqrt{\epsilon})n &\geq (\max\{1.5\alpha_1 + 0.5\alpha_3, \alpha_3\} + 9\sqrt{\epsilon})n \\ &\geq (1.5\alpha_1 + 0.5\alpha_2 + 9\sqrt{\epsilon})n \end{aligned}$$

vertices. If G' is non-bipartite we are done again, if not then one can apply Lemma 14 to find a monochromatic component with large matching in one of the first two colors.

Thus, we have showed that

$$\begin{aligned} R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle}) \\ \leq (\max\{2\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 0.5\alpha_1 + 0.5\alpha_2 + \alpha_3\} + o(1))n. \end{aligned}$$

In order to complete the proof of Theorem 1 we need to specify colorings which result in the matching lower bound.

Again we may and shall assume that $\alpha_1 \geq \alpha_2$. Let us consider the complete graph on

$$N = 2\langle\alpha_1 n\rangle + \langle\alpha_2 n\rangle - 4$$

vertices whose vertex set is partitioned into four disjoint sets V_1, V_2, V_3, V_4 , where $|V_1| = |V_2| = \langle\alpha_1 n\rangle - 1$ and $|V_3| = |V_4| = 0.5\langle\alpha_2 n\rangle - 1$. Let us color all edges contained in one of the sets V_i 's with the first color, all edges joining V_1 and V_3 , and those joining V_2 and V_4 , with the second color, and all other edges with the third color. It is easy to see that this coloring gives neither a cycle longer than $\langle\alpha_1 n\rangle - 1$ in the first color, nor a cycle longer than $\langle\alpha_2 n\rangle - 2$ in the second color. It also leads to no odd cycles in the third color. Consequently,

$$R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle}) \geq 2\langle\alpha_1 n\rangle + \langle\alpha_2 n\rangle - 3.$$

Finally, let us consider the complete graph on

$$\bar{N} = 0.5\langle\alpha_1 n\rangle + 0.5\langle\alpha_2 n\rangle + \langle\alpha_3 n\rangle - 3$$

vertices whose set of vertices \bar{V} is split into three parts $\bar{V}_1, \bar{V}_2, \bar{V}_3$, where $|\bar{V}_1| = 0.5\langle\alpha_1 n\rangle - 1$, $|\bar{V}_2| = 0.5\langle\alpha_2 n\rangle - 1$, and $|\bar{V}_3| = \langle\alpha_3 n\rangle - 1$. Let us color edges of $K_{\bar{N}}$ by coloring all edges contained in \bar{V}_3 with the third color, all edges with at least one end in \bar{V}_2 with the second color, and all other edges with the first color. In this coloring there are no cycles longer than $\langle\alpha_1 n\rangle - 2$ in the first color, no cycles longer than $\langle\alpha_2 n\rangle - 2$ in the second color, and no cycles longer than $\langle\alpha_3 n\rangle - 1$ colored in the third color. Thus,

$$R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle}) \geq 0.5\langle\alpha_1 n\rangle + 0.5\langle\alpha_2 n\rangle + \langle\alpha_3 n\rangle - 2.$$

This completes the proof of Theorem 1(ii). \square

6. TRIPLE OF CYCLES: TWO ODD, ONE EVEN

In this section we shall complete the proof of Theorem 1, showing the estimates for $R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle})$.

Proof of Theorem 1(iii). The proof of the upper bound for $R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle})$ is, again, based on Lemma 3, which states that it is enough to check that for $\alpha_1, \alpha_2, \alpha_3 > 0$, where $c = \max\{4\alpha_1, \alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_3\}$, we have $\tau_{1,2}(\alpha_1, \alpha_2, \alpha_3; c)$ holds. Note that we may and shall assume that $\alpha_2 = \alpha_3$, and $\max\{\alpha_1, \alpha_2\} = 1$.

Thus, for a small $\epsilon > 0$ consider a graph $G = (V, E)$ obtained from the complete graph on $N = (c + 15\sqrt{\epsilon})n$ vertices by removing at most $|E| \geq \epsilon^5 n^2$ edges. Let G_1, G_2, G_3 be a partition of G induced by a three-coloring of its edges. If the average degree $d(G_1)$ of G_1 is larger than $(\alpha_1 + \epsilon)n$ then, by Theorem 8, it contains also a cycle longer than $\alpha_1 n + 1$ and so we are done. Thus, let us assume that $d(G_1) < (\alpha_1 + \epsilon)n$. Observe that $c \geq \alpha_1 + 2\alpha_2$, so in this case one of the graphs G_2 and G_3 , say, G_2 , has the average degree larger than

$$\theta n = 0.5|V| - (\alpha_1 + \epsilon)n \geq (\alpha_2 + 7\sqrt{\epsilon})n.$$

Suppose that G_2 contains no non-bipartite component saturating at least $(\alpha_2 + \epsilon)n$ vertices. Then, one can apply to G_2 Lemma 9 and decompose it into two subgraphs G' and G'' , where G' is bipartite with bipartition (X, Y) , $|X| \geq |Y|$ and $d(G'') \leq \alpha_2 n$. Since $d(G'') \leq \alpha_2 n < d(G_2)$, we must have $d(G') \geq d(G_2) \geq \theta n$, and so $|X| + |Y| \geq 2\theta n$, and $|X| \geq \theta n$. Let us consider the graph $H = (V', E')$ with the vertex set $V \setminus Y$ whose edges are all edges of G which are colored with the first or the third color and either are contained in X , or join X to $V \setminus Y$. Then H is ‘nearly complete’ graph with the hole $W = V \setminus (X \cup Y)$, $|W| = \nu n$, colored with two colors. We shall show that H fulfills assumptions of Lemma 13 so that it contains either a component in the first color containing a matching saturating at least $\alpha_1 n$ vertices, or a non-bipartite component in the third color with a matching saturating at least $\alpha_3 n$ vertices.

Let us consider two cases.

Case 1. $\alpha_2 \leq 1.5\alpha_1$.

In this case we have $|V| = (4\alpha_1 + 15\sqrt{\epsilon})n$, $\theta \geq 1.5\alpha_1 + 7\sqrt{\epsilon}$,

$$\nu \leq |V|/n - 2\theta \leq \alpha_1 + \sqrt{\epsilon}$$

and

$$|V'| \geq \frac{|V| - \nu n}{2} + \nu n = (2\alpha_1 + 0.5\nu + 7\sqrt{\epsilon})n.$$

Thus, we have to verify that if $\alpha_2 \leq 1.5\alpha_1$, and $\nu \leq \alpha_1 + \sqrt{\epsilon}$, then for the function $\xi(\alpha_1, \alpha_2, \nu)$ defined by (8), i.e.

$$\xi(\alpha_1, \alpha_2, \nu) = \max\{\alpha_1 + 0.5\alpha_2, 0.5\alpha_1 + \alpha_2, \\ 0.5\alpha_1 + 0.5\alpha_2 + \nu, 2\alpha_1, 1.5\alpha_1 + \nu\},$$

we have

$$\xi(\alpha_1, \alpha_2, \nu) \leq 2\alpha_1 + 0.5\nu + \sqrt{\epsilon}.$$

However, the above fact follows from the definition of $\xi(\alpha_1, \alpha_2, \nu)$ and the following five inequalities:

$$\begin{aligned} \alpha_1 + 0.5\alpha_2 &\leq \alpha_1 + 0.75\alpha_1 \leq 2\alpha_1 + 0.5\nu, \\ 0.5\alpha_1 + \alpha_2 &\leq 2\alpha_1 \leq 2\alpha_1 + 0.5\nu, \\ 0.5\alpha_1 + 0.5\alpha_2 + \nu &= 0.5\alpha_1 + 0.75\alpha_1 + 0.5\nu + 0.5\nu \\ &\leq 2\alpha_1 + 0.5\nu + \sqrt{\epsilon}, \\ 2\alpha_1 &\leq 2\alpha_1 + 0.5\nu, \\ 1.5\alpha_1 + \nu &= 1.5\alpha_1 + 0.5\nu + 0.5\nu \leq 2\alpha_1 + 0.5\nu + \sqrt{\epsilon}. \end{aligned}$$

Thus, the existence of a monochromatic component which contains a large matching in either the first or the second color follows from Lemma 13.

Case 2. $\alpha_2 \geq 1.5\alpha_1$.

Here $|V| = (\alpha_1 + 2\alpha_2 + 15\sqrt{\epsilon})n$, $\theta \geq \alpha_2 + 7\sqrt{\epsilon}$, $\nu \leq \alpha_1 + \sqrt{\epsilon} \leq \frac{2}{3}\alpha_2 + \sqrt{\epsilon}$, and

$$|V'| \geq \frac{|V| - \nu n}{2} + \nu n = (0.5\alpha_1 + \alpha_2 + 0.5\nu + 7\sqrt{\epsilon})n.$$

Again, we check that in this case

$$\xi(\alpha_1, \alpha_2, \nu) \leq 0.5\alpha_1 + \alpha_2 + 0.5\nu + \sqrt{\epsilon}, \quad (12)$$

by the direct inspection:

$$\begin{aligned}
 \alpha_1 + 0.5\alpha_2 &\leq 0.5\alpha_1 + \frac{5}{6}\alpha_2 \leq 0.5\alpha_1 + \alpha_2 + 0.5\nu, \\
 0.5\alpha_1 + \alpha_2 &\leq 0.5\alpha_1 + \alpha_2 + 0.5\nu, \\
 0.5\alpha_1 + 0.5\alpha_2 + \nu &= 0.5\alpha_1 + 0.5\alpha_2 + 0.5\nu + 0.5\nu \\
 &\leq 0.5\alpha_1 + \alpha_2 + 0.5\nu + \sqrt{\epsilon}, \\
 2\alpha_1 &\leq 0.5\alpha_1 + \alpha_2 + 0.5\nu, \\
 1.5\alpha_1 + \nu &\leq \alpha_2 + 0.5\nu + 0.5\nu \leq 0.5\alpha_1 + \alpha_2 + 0.5\nu + \sqrt{\epsilon}.
 \end{aligned}$$

Now we can employ Lemma 13 to complete the proof of the lower bound for $R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle})$.

In order to show the lower bound for $R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle})$ let us observe that the same coloring we have employed for estimating the Ramsey number for three odd cycles in the proof of Theorem 1(iv) can be used to show that

$$R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle}) \geq 4\langle\alpha_1 n\rangle - 3.$$

Finally, let us consider the complete graph on

$$\tilde{N} = \langle\alpha_1 n\rangle + 2\langle\alpha_2 n\rangle - 4$$

vertices whose vertex set is partitioned into four parts $\tilde{V}_1, \tilde{V}_2, \tilde{V}_3, \tilde{V}_4$, such that $|\tilde{V}_1| = |\tilde{V}_2| = 0.5\langle\alpha_1 n\rangle - 1$, and $|\tilde{V}_3| = |\tilde{V}_4| = \langle\alpha_2 n\rangle - 1$. Let us color all edges of $K_{\tilde{N}}$ contained in either \tilde{V}_1 or \tilde{V}_2 , with the first color, and use the same color to color all edges between the pairs \tilde{V}_1 and \tilde{V}_3 , and between \tilde{V}_2 and \tilde{V}_4 , all edges contained in either \tilde{V}_3 or \tilde{V}_4 we color with the second color, and all other edges with the third color. It can be easily seen that in this coloring no cycle longer than $0.5\langle\alpha_1 n\rangle - 2$ is colored with the first color, no cycle longer than $\langle\alpha_2 n\rangle - 1$ is colored with the second color, and no odd cycle is colored with the third color. Consequently,

$$R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle}) \geq \langle\alpha_1 n\rangle + 2\langle\alpha_2 n\rangle - 3.$$

An analogous construction gives

$$R(C_{\langle\alpha_1 n\rangle}, C_{\langle\alpha_2 n\rangle}, C_{\langle\alpha_3 n\rangle}) \geq \langle\alpha_1 n\rangle + 2\langle\alpha_3 n\rangle - 3.$$

This completes the proof of (iii) and Theorem 1. \square

REFERENCES

1. J.A. Bondy and P. Erdős, *Ramsey numbers for cycles in graphs*, J. Combin. Theory **14** (1973), 46–54.
2. P. Erdős, *On the combinatorial problems which I would most like to see solved*, Combinatorica **1** (1981), 25–42.

3. P. Erdős, R.J. Faudree, C.C. Rousseau, and R.H. Schelp, *Generalized Ramsey theory for multiple colors*, J. Combin. Th., Series B, **20** (1976), 250–264.
4. P. Erdős and T. Gallai, *On maximal paths and circuits of graphs*, Acta Math. Acad. Sci. Hungarica **10** (1956), 337–356.
5. R.J. Faudree and R.H. Schelp, *All Ramsey numbers for cycles in graphs*, Discrete Math. **8** (1974), 313–329.
6. A. Figaj and T. Łuczak, *The Ramsey number for a triple of long even cycles*, J. Combin. Th. Series B **97** (2007), 584–596.
7. A. Gyárfás, M. Ruszinkó, G.N. Sárközy, and E. Szemerédi, *Three-color Ramsey numbers for paths*, Combinatorica **27** (2007), 35–69.
8. G. Károlyi and V. Rosta, *Generalized and geometric Ramsey numbers for cycles*, Theoret. Comp. Sci., **263** (2001), 87–98.
9. J. Komlós and M. Simonovits, *Szemerédi’s Regularity Lemma and its applications in graph theory*. In “Combinatorics, Paul Erdős is Eighty” (D. Miklós, V. T. Sós, T. Szőnyi, Eds.) Bolyai Society Mathematical Studies, vol.2, Budapest, 1996, 295–352.
10. Y. Kohayakawa, M. Simonovits, and J. Skokan, *The 3-colored Ramsey number of odd cycles*, Proceedings of GRACO2005, Electron. Notes Discrete Math., 19, Elsevier, Amsterdam, 2005, 397–402 (electronic).
11. T. Łuczak, $R(C_n, C_n, C_n) \leq (4 + o(1))n$, J. Combin. Th., Series B **75** (1999), 174–187.
12. S.P. Radziszowski, *Small Ramsey numbers*, Electronic J. Combin. (2002), DS1.9., 1–42.
13. V. Rosta, *On a Ramsey type problem of J.A. Bondy and P. Erdős, I & II*, J. Combin. Th., Series B **15** (1973), 94–120.
14. E. Szemerédi, *Regular partitions of graphs*. In “Problèmes Combinatoires et Théorie des Graphes, Proc. Colloque Inter. CNRS” (J.-C. Bermond, J.-C. Fournier, M. Las Vergnas, et D. Sotteau, Eds.), CNRS, Paris, 1978, 399–401.

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